Analysis I Lecture 25

Last time: Integration tools:

- Substitution
- Integration by parts
- Integrating rational functions

Theorem 8.29 (Substitution or change of variables)

Let $f : [a,b] \to \mathbb{R}$ continuous and

$\varphi : [\alpha,\beta] \longrightarrow [a,b]$ be a C^1-function then

$$\boxed{\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} \underline{f(\varphi(t)) \cdot \varphi'(t)}\,dt}$$

substitution $x = \varphi(t)$

One way to remember: $\Rightarrow dx \rightsquigarrow d\varphi(t)$

Apply diff.

to move here
Apply different

$$\int f(x)\, \underline{\underline{dx}} = \int f(\varphi(t))\, d\varphi(t) =$$

We want to substitute $x = \varphi(t)$

$$= \int f(\varphi(t)) \cdot \varphi'(t)\, dt$$

Theorem 8.34 (Integration by parts)

Let $f, g : [a,b] \to \mathbb{R}$ be two C^1-functions

Then

$$\boxed{\int_a^b f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b f'(x) \cdot g(x)\, dx}$$

Alternatively one can write:

$$\boxed{\int_a^b f(x)\cdot \underbrace{dg(x)}_{\substack{\shortparallel \\ g'(x)\cdot dx}} = f\cdot g\Big|_a^b - \int_a^b g(x)\, \underbrace{d f(x)}_{\substack{\shortparallel \\ f'(x)\cdot dx.}}}$$

Integrating rational functions:

Main tool is to use partial fractions decomposition:

$\deg P_i < \deg Q_i$

For $\deg P < \deg Q$

$$\frac{P(x)}{\underbrace{Q_1(x)\cdot Q_2(x)}} = \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)}$$

What are the smallest Q_i's we can get?

Fundamental theorem of algebra $/\mathbb{R}$

Any polynomial $F(x)$ with real coefficients can be written as a product:

$$F(x) = d \cdot L_1^{k_1} \cdots L_s^{k_s} \cdot Q_1^{\ell_1} \cdots Q_t^{\ell_t}$$

where $L_i = (x - r_i)$ and $Q_i = (1 + b_i x + c_i x^2)$

Proposition 8.43

So any rational function can be written as:

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\alpha L_1^{k_1} \cdots L_s^{k_s} \cdot Q_1^{\ell_1} \cdots Q_r^{\ell_r}} =$$

$$= \alpha_1 R_1 + \cdots + \alpha_t R_t$$ where

$R_i(x)$ is either 1) polynomial ← appears if $\deg P \geq \deg Q$.

2) $\dfrac{1}{(x-r)^p}$ 3) $\dfrac{x+c}{(x^2+2rx+s)^p}$

Example $\displaystyle\int \frac{1}{x^3-4x^2+9x-10}\,dx$

1st Let's find a real root of $x^3-4x^2+9x-10 = f(x)$

If $f(x)$ has natural root then it devides $-\frac{10}{1} = -10$

$\rightsquigarrow$ So it is $\pm 1 \quad \pm 2 \quad \pm 5 \quad \pm 10$

In our case $f(2)=0$:

So $x^3-4x^2+9x-10$ is divisible

by $x-2$:

$$
\begin{array}{rrrr|l}
x^3 & -4x^2 & +9x & -10 & x-2 \\
\cline{5-5}
-(x^3 & -2x^2) & & & x^2-2x+5 \\
\cline{2-4}
 & -2x^2 & +9x & -10 & \\
 & -(-2x^2 & +4x) & & \\
\cline{3-4}
 & & 5x & -10 & \\
 & & -(5x & -10) & \\
\cline{3-4}
 & & & 0 &
\end{array}
$$

So we get

$$x^3 - 4x^2 + 9x - 10 = (x-2)\underbrace{(x^2 - 2x + 5)}_{\text{doesn't have real roots.}}$$

Want to write:

$$\frac{1}{x^3 - 4x^2 + 9x - 10} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 5}$$

for some A, B, C

$$\frac{1}{x^3-4x^2+9x-10} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+5} =$$

$$= \frac{A(x^2-2x+5) + (x-2)(Bx+C)}{(x-2)(x^2-2x+5)} =$$

$$= \frac{(A+B)x^2 + (C-2A-2B)x + (5A-2C)}{(x-2)(x^2-2x+5)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C-2A-2B=0 \\ 5A-2C=1 \end{cases} \Leftrightarrow \begin{cases} A=\frac{1}{5} \\ B=-\frac{1}{5} \\ C=0 \end{cases} \Rightarrow$$

$$\frac{1}{x^3-4x^2+9x-10} = \frac{1}{5}\cdot\frac{1}{x-2} - \frac{1}{5}\cdot\frac{x}{x^2-2x+5}$$

$$\Rightarrow \int_3^5 \frac{1}{x^3-4x^2+9x-10}\,dx = \frac{1}{5}\int_3^5 \frac{1}{x-2}\,dx - \frac{1}{5}\int \frac{x}{x^2-2x+5}\,dx$$

$$\boxed{\frac{1}{5}\log|x-2|\Big|_3^5}$$

$$\int_3^5 \frac{x}{x^2-2x+5}\,dx$$

$$\arctan' x = \frac{1}{1+x^2}$$

$$\frac{x}{(x^2-2x+1)-1+5} = \frac{x}{(x-1)^2+4} =$$

$$= \frac{1}{4}\cdot\frac{x}{\frac{(x-1)^2}{4}+1} = \frac{1}{4}\cdot\frac{x}{\left(\frac{x-1}{2}\right)^2+1}$$

$$\int_3^5 \frac{x}{x^2-2x+5}\,dx = \frac{1}{4}\int_3^5 \frac{x}{\left(\frac{x-1}{2}\right)^2+1}\,dx =$$

Substitution

$$\frac{x-1}{2} = y \qquad x = 2y+1$$

$$dx = 2\,dy$$

$$x = 3 \rightarrow y = 1$$

$$x = 5 \rightarrow y = 2$$

$$= \frac{1}{4}\int_1^2 \frac{2y+1}{y^2+1}\,2\,dy =$$

$$= \frac{1}{4}\int_1^2 \frac{2y+1}{y^2+1}\, 2\, dy = \underbrace{\frac{1}{2}\int_1^2 \frac{2y}{y^2+1}\, dy}_{} +$$

$$+ \underbrace{\frac{1}{2}\int_1^2 \frac{1}{y^2+1}\, dy}_{}$$

$$\boxed{= \frac{1}{2} \arctan(y) \Big|_1^2}$$

using substitution

$t = y^2+1$ we get $\boxed{\log(y^2+1)\Big|_1^2}$

Summing everything together we get

$$\int_3^5 \frac{1}{x^3 - 4x^2 + 9x - 10} = \frac{1}{5} \log |x-2| \Big|_3^5 +$$

$$- \frac{1}{10} \log(y^2+1) \Big|_1^2 -$$

$$\arctan(y) \Big|_1^2 .$$

Once rational function is written in this form it is enough to know how to integrate each one of them:

1) $$\int \frac{1}{(x-r)^p}\,dx = \begin{cases} \log|x-r| & p=1 \\ \dfrac{1}{(1-p)\cdot(x-r)^{p-1}} & p>1 \end{cases}$$

$$(x^n)' = n\cdot x^{n-1}$$

$$\Rightarrow \left(\frac{1}{x^{p-1}}\right)' = \left(x^{(1-p)}\right)' = (1-p)\cdot x^{-p} = \frac{1-p}{x^p}$$

2) $$\int \frac{x+c}{(x^2+2rx+s)^p}\,dx =$$

$$x+c = (x+r)+c-r$$

$$= \int \frac{(x+r)}{(x^2+2rx+s)^p}\,dx + \boxed{\int \frac{c-r}{(x^2+2rx+s)^p}\,dx}$$

$$u = x^2+2rx+s \rightsquigarrow u' = 2x+2r$$

So we get

$$\int \frac{x+r}{(x^2+2r+s)^p}\,dx = \frac{1}{2}\int \frac{1}{u^p}\,du =$$

$$u = x^2+2r+s$$

$$= \begin{cases} \log|u| & \text{if } p=1 \\ \dfrac{1}{(1-p)u^{p-1}} & \text{if } p>1 \end{cases}$$

Last piece:

$$\int \boxed{\frac{1}{(x^2+2rx+s)^p}\, dx}$$ by substitution

we can bring it to the form:

$$I_p = \int \frac{1}{(u^2+1)^p}\, du$$

Moreover using integrating by parts we get:

$$\boxed{I_{p+1} = \frac{\frac{u}{(u^2+1)^p} + (2p-1) I_p}{2p}}$$

with $I_1 = \arctan(u)$.

Bonus: Rational functions in exponentials and roots.

Example $\displaystyle\int \frac{1}{e^x+1}\,dx = \int \underbrace{\frac{1}{(e^x+1)e^x}}\,e^x\,dx$

want to use $u = e^x$

$du = \underbrace{e^x\,dx}$

$$= \int \frac{1}{(u+1)u}\,du = \int \frac{1}{u} - \frac{1}{u+1}\,du$$

$$= \int \frac{1}{u} - \frac{1}{u+1}\, du = \log|u| - \log|u+1| =$$

$$= \log(e^x) - \log(e^x+1) =$$

$$= x - \log(e^x+1).$$

Can do the same with $\sqrt{x}$

Want $u = \sqrt{x}$ then

$$du = \underbrace{\frac{1}{2\sqrt{x}}\, dx}.$$

$\underline{\text{E.g.}}$ $\displaystyle \int \frac{1}{\sqrt{x}+1}\, dx = \int \underbrace{\frac{1}{\sqrt{x}+1} \cdot 2\sqrt{x}}\, \cdot \underbrace{\frac{1}{2\sqrt{x}}\, dx}_{du}$

$$= \int \frac{2u}{u+1}\, du = \int 2\, du - \int \frac{2}{u+1} du$$

$$\int 2\,du + \int \frac{2}{u+1}\,du = 2u - 2\log|u+1| =$$

$$= 2\sqrt{x} - 2\log(\sqrt{x}+1).$$

Partial fractions are useful outside of integration:

E.g Compute Taylor expansion of

$$\frac{1}{\underbrace{(1+x)(1-x)}_{=(1-x^2)}}$$

at $x_0 = 0$.

Idea: Use partial fractions

$$\frac{1}{(1+x)(1-x)} = \underline{\frac{1}{2}\left(\frac{1}{x+1}\right)} + \underline{\frac{1}{2}\left(\frac{1}{1-x}\right)}.$$

So we get

$$T_{f,0} = \frac{1}{2} T_{\frac{1}{1-x},0} + \frac{1}{2} T_{\frac{1}{1+x},0} =$$

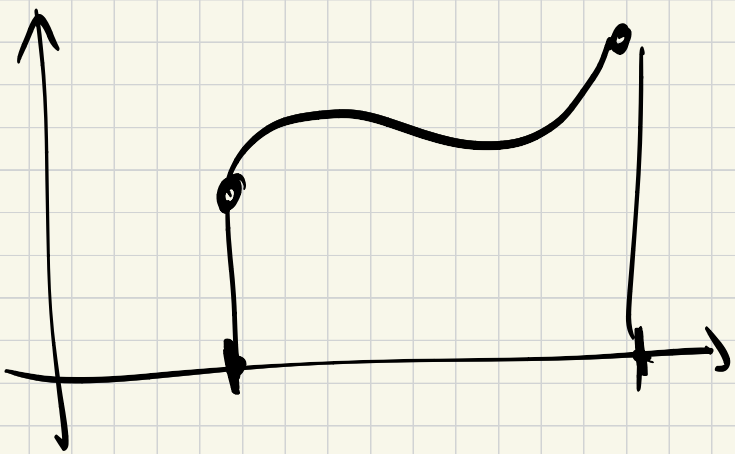
$$= \frac{1}{2} \left(\sum_{i=0}^{\infty} x^i + \sum_{i=0}^{\infty} (-1)^i x^i \right) =$$

$$= \sum_{i=0}^{\infty} x^{2i} = \sum_{i=0}^{\infty} (x^2)^i = \underbrace{\frac{1}{1-x^2}}_{} \blacksquare$$

Today:

Improper integrals.

<u>Motivation</u> So far we integrated continuous functions over closed intervals:



What is $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ or $\displaystyle\int_0^1 \log(x)\,dx$

Definition 8.52

Let $f: I \to \mathbb{R}$ be a continuous function then

1) If $I = [a, b)$ with $b \in \mathbb{R}$ or $+\infty$ we define improper integral to be:

Notation

$$\int_a^{b_-} f(t)\,dt := \lim_{\substack{x \to b^- \\ =}} \Big(\underbrace{\int_a^x f(t)\,dt}_{} \Big)$$

since f is continuous f is integrable between a and x.

provided the limit exists

2) If $I = (a, b]$ with $a \in \mathbb{R}$ or $-\infty$ we define improper integral to be:

$$\int_{a_+}^{b} f(t)\,dt := \lim_{x \to a^+} \left(\int_x^b f(t)\,dt \right)$$

provided the limit exists.

3) If $I = (a, b)$ with $a \in \mathbb{R}$ or $-\infty$, $b \in \mathbb{R}$ or $+\infty$

we define improper integral to be:

$$\int_{a^+}^{b^-} f(x)\,dx = \underbrace{\int_{a^+}^{c} f(x)\,dx} + \underbrace{\int_{c}^{b^-} f(x)\,dx}$$

for any $c \in I$ provided that both of them exist.

Often people drop + and - in the notation of improper integral!

Instead $\int_{0^+}^{1} \log(t)\, dt$ we often write $\int_{0}^{1} \log(t)\, dt$.

Examples $\int_1^\infty \frac{1}{x^2}\,dx =$

$$= \lim_{R\to\infty} \underbrace{\int_1^R \frac{1}{x^2}\,dx} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^R =$$

$$= \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = 1.$$